THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-first regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 6. There were present eighteen persons, including the following twelve members of the Society:

Professor R. E. Allardice, Professor Harry Bateman, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Dr. Pauline Sperry, Professor R. M. Winger.

Professor W. A. Manning occupied the chair. At the close of the session attending members were entertained at luncheon, and in the afternoon an interesting demonstration of experiments on aeroplane propellers was given by Professors Moreno and Lesley, of Stanford University.

It was provisionally decided that the next meeting at Stanford University should be held on April 5, 1919.

The following papers were read at this meeting:

(1) Professor D. N. LEHMER: "On Jacobi's extension of the continued fraction algorithm" (preliminary report).

(2) Professor H. F. BLICHFELDT: "A second principle in the geometry of numbers."

(3) Professor M. W. HASKELL: "Continuous groups of quadratic transformations."

(4) Professor D. N. LEHMER: "Arithmetical theory of certain Hurwitzian continued fractions."

(5) Professor E. T. BELL: "Numerical functions of $[x]$."

(6) Professor E. T. BELL: "Fourier series for certain elliptic functions."

(7) Professor R. M. WINGER: "On the satellite line of the cubic."

(8) Professor R. M. WINGER: "Self-projective rational septimics."

Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. Professor Lehmer developed new connections between

the ordinary continued fraction and the extension made by Jacobi in 1868, with applications to the theory of numbers and to the theory of equations.

2. In a former paper (*Transactions of the American Mathematical Society*, 1914, pages 227–235) Professor Blichfeldt proved a geometrical theorem having a number of applications in the theory of numbers; for instance, the variables of a positive-definite quadratic form f in n variables and of determinant D can be given such integral values, not all zero, that the numerical value of f is not greater than a number which has $(n/\pi e)D^{1/n}$ for its asymptotic value. This limit being too high, the question arises what may the actual lowest limit be; i. e., there is need of a theorem giving a limit which is exceeded by at least *one* quadratic form. Minkowski has given such a theorem in geometrical form (Collected Works, volume 2, page 270) without proof, namely: an oval surface (like an ellipse) in space of n dimensions, having a center at the origin and a volume $\leqq 2(1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots)$, can be deformed by a homogeneous linear transformation which preserves volumes, into a surface which contains no lattice points except the origin. Professor Blichfeldt states a theorem which gives somewhat better results than that of Minkowski; thus, applied to ellipsoids, this theorem is that the volume that may be assumed approaches a number $\geqq 23$ for large values of n (instead of 2 by Minkowski's theorem).

3. Professor Haskell has examined the various types of continuous transformations of the plane which in homogeneous form are rational quadratic transformations. They are all six-parameter groups of mixed quadratic and linear transformations, with two-parameter subgroups which are purely quadratic. Besides those which are simple products of linear transformations of the separate variables, they may be classified as elliptic, hyperbolic, and parabolic—the elliptic type including as a special case the well known inversion group.

4. If A_m/B_m is the mth convergent of the Hurwitzian continued fraction

$$\overline{(a_1, a_2, a_3, \cdots, a_{k-1}, ma_k)} \qquad (m = 1, 2, 3, \cdots)$$

and A_m'/B_m' that of

$$\overline{(a_{k-1}, a_{k-2}, \cdots, a_2, a_1, -ma_k - 2M)}, \quad (m = 1, 2, 3, \cdots)$$

and A_m''/B_m'' the $(m+1)$st convergent of

$$\overline{(x, a_{k-1}, a_{k-2}, \cdots, a_2, a_1, y, a_{k-1}, a_{k-2}, \cdots, a_2, a_1, -ma_k - 2M)}, \quad (m = 2, 3, \cdots)$$

where

$$M = \frac{B_{k-1} + A_{k-2}}{A_{k-1}}, \quad x = \frac{B_{k-1} - A_{k-2}}{A_{k-1}},$$

$$y = -A_k - 2\left(\frac{B_{k-1}^2 + B_{k-2}A_{k-1}}{A_{k-1}B_{k-1}}\right)$$

and where $a_1, a_2, a_3, \cdots, a_{k-1}$ are positive or negative integers or zero, and a_k, A_{k-1}, B_{k-1} are positive or negative integers not zero, then Professor Lehmer shows

$$A_{\rho k-1} = (-1)^{\rho-1}A'_{\rho k-1},$$
$$A_{\rho k} = (-1)^{\rho}(A'_{\rho k} + MA'_{\rho k-1}),$$
$$B_{\rho k-1} = (-1)^{\rho-1}A''_{\rho k-1},$$
$$B_{\rho k} = (-1)^{\rho}(A''_{\rho k} + MA''_{\rho k-1}).$$

From these equations he develops the arithmetical theory of such continued fractions, and shows, among other things, that for n prime to a_k, A_{k-1}, B_{k-1}, we have always

$$A_{2nk-1} \equiv B_{2nk} \equiv 0 \pmod{n},$$
$$A_{2nk} \equiv B_{2nk-1} \equiv (-1)^{kn-1} \pmod{n}.$$

In particular the denominators of the convergents of orders $3n$, $3n-2$, and $3n-6$, and the numerator of that of order $3n-3$ for the regular continued fraction which represents the base of Naperian logarithms are always divisible by n.

5. Simple methods are given by Professor Bell for finding relations between, and reducing, sums of numerical functions of $[x]$, the greatest positive integer contained in x. Such reductions are useful in the investigation of arithmetical mean values, etc. The paper appeared in part in the March number of the *Annals of Mathematics*.

6. In Professor Bell's second paper the series are for the squares and certain products of the complete set of eighteen doubly periodic theta quotients considered by Hermite in his memoir on Kronecker's class-number relations. The series have important arithmetical consequences. The paper has appeared in the *Messenger of Mathematics*.

7. While Salmon discusses the satellite line of the cubic at some length he fails to give its equation. The symbolical expression for this important covariant has been supplied by Morley. In the present paper Professor Winger derives the explicit equation of the satellite, both for the rational and general cubic, in canonical forms, and discusses associated loci. Several chain theorems are obtained and a generalization is made for the plane curve of order n.

8. Professor Winger's second paper, a preliminary report of which was made to the Society at a former meeting, is intended as a sequel to two papers of similar titles which have already appeared in the *American Journal of Mathematics*. The varieties of self-projective rational septimics in parametric form are exhibited and the more striking properties of certain of the curves are pointed out. According to the criterion of classification adopted, there are 26 distinct types, invariant under cyclic groups of order 2, 3, 4, 5, 6, and 7, dihedral groups of order 6, 10, and 14, and an infinite group.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE tenth regular meeting of the American Mathematical Society at Chicago, being also the forty-first regular meeting of the Chicago Section, was held on Friday and Saturday, April 12 and 13, at the University of Chicago. The various sessions were attended by about fifty persons, among whom were the following thirty-five members of the Society:

Professor R. P. Baker, Professor G. A. Bliss, Professor J. W. Bradshaw, Professor R. D. Carmichael, Professor A. R.

Crathorne, Mr. G. H. Cresse, Professor L. E. Dickson, Professor Arnold Dresden, Professor J. W. Glover, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor A. M. Kenyon, Professor S. Lefschetz, Professor A. C. Lunn, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor C. N. Moore, Professor E. H. Moore, Professor R. E. Moritz, Professor E. J. Moulton, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. G. W. Smith, Mr. L. L. Steimley, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young.

Forty-two persons joined in a dinner at the Quadrangle Club on Friday evening, which was followed by a number of short informal speeches, President Dickson presiding.

At the session of Friday forenoon, Professor Hedrick read a paper in memory of Professor E. W. Davis, whose death occurred on February 3, 1918. A motion to transmit a copy of Professor Hedrick's paper to the family of Professor Davis was made by Professor Slaught and concurred in by a rising vote of the meeting.

Friday afternoon was devoted to a symposium on "Divergent series and summability." The principal papers were presented by Professor R. D. Carmichael, who spoke on "General aspects of the theory of summable series," and by Professor C. N. Moore, whose subject was "Applications of the theory of summability to developments in orthogonal functions." These papers were followed by a general discussion by Professors E. H. Moore, Hedrick, Curtiss, Hildebrandt, Bliss, and others.

Professor Bliss, chairman of the section, presided at the meetings on Friday; Professor Weld presided on Saturday morning.

At the sessions of Friday forenoon and Saturday forenoon, the following papers were read:

(1) Professor C. N. MOORE: "On the summability of the developments in Bessel's functions."

(2) Mr. M. G. SMITH: "On the zeros of functions defined by linear homogeneous differential equations, containing a parameter."

(3) Professor C. H. SISAM: "On surfaces containing a system of cubics which do not constitute a pencil."

(4) Mr. E. P. LANE: "Conjugate systems with indeterminate axis curves."

(5) Mr. R. F. BORDEN: "On the Laplace-Poisson mixed equation."

(6) Professor R. D. CARMICHAEL: "On a general class of integrals of the form $\int_0^\infty \varphi(t)g(x+t)/g(x)dt$."

(7) Professor S. LEFSCHETZ: "Abelian manifolds."

(8) Dr. P. R. RIDER: "On the f_1 function of the calculus of variations."

(9) Professor E. J. WILCZYNSKI: "Some geometric aspects of the theory of functions."

(10) Mr. L. J. ROUSE: "A contribution to the question of linear dependence in linear integral equations."

(11) Professor G. A. MILLER: "Determinant groups."

(12) Professor G. A. MILLER: "Group theory proof of two elementary theorems in number theory."

(13) Professor LOUIS BRAND: "Flexural deflections and statically indeterminate beams."

(14) Professor ARNOLD EMCH: "Proof of Pohlke's theorem and its generalization by affinity."

(15) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group. Second paper."

(16) Dr. A. R. SCHWEITZER: "Concerning independence."

(17) Professor T. H. HILDEBRANDT: "On a generalization of a theorem of Toeplitz."

Mr. Smith and Mr. Borden were introduced by Professor Carmichael, Mr. Lane by Professor Wilczynski, and Mr. Rouse by Professor Hildebrandt. The papers of Dr. Rider, Mr. Rouse, Professor Miller, Professor Brand and Professor Emch were read by title.

Abstracts of the papers follow, the numbers corresponding to those in the list above.

1. The principal object of Professor Moore's paper is to extend certain results of a paper presented to the Society at the last summer meeting (see this BULLETIN, volume 24, page 65). In the previous paper the summability $(C, k > \frac{1}{2})$ at the origin, and the uniform summability $(C, k > \frac{1}{2})$ in the neighborhood of the origin, of the Bessel's development of an arbitrary function satisfying certain conditions, were established. In the present paper it is further

shown that the development is summable $(C, \frac{1}{2})$ at the origin and uniformly summable $(C, \frac{1}{2})$ in its neighborhood, under the same conditions on the arbitrary function. It is also shown that this is all that can be established for functions of the type considered, as an example is given of such a function for which the development is non-summable $(C, 0 \leqq k < \frac{1}{2})$ at the origin.

2. Using a certain generalization of a theorem due to Professor R. D. Carmichael and published in the *Annals of Mathematics* for March, 1918, Mr. Smith has derived conditions under which one can be assured that any solution of a differential equation of general order n has at least a given number of zeros in a fixed (finite) interval. It is found that the theorems are most easily applied in the case of an equation of even order of the form

$$E(u) \equiv \frac{d^{2m}}{dx^{2m}}[e_{2m}(x, \lambda)u] + \lambda^2 \frac{d^{2m-2}}{dx^{2m-2}}[e_{2m-2}(x, \lambda)u] + \cdots + \lambda^{2m} e_0(x, \lambda)u = 0.$$

For the particular case of $2m = 4$, one of the theorems takes the following form: If the parameter λ in the equation $E(u) = 0$ is fixed and satisfies the relations $\lambda \geqq S$, $\lambda \geqq s\pi/(b - a)$, and if for every x in the interval $a \leqq x \leqq b$ we have

$$e_0(x, \lambda) \geqq 3e_2 - 21e_4 + (60e_4 - 6e_2) \tan^2 \lambda (x - a + \pi/2\lambda),$$

the equality sign not holding throughout (a, b), then any solution $u(x, \lambda)$ of $E(u) = 0$ has at least one zero in each subinterval of (a, b) of length π/λ measuring from the point a, and hence has at least s zeros in (a, b).

3. Professor Sisam's paper classifies completely the types of algebraic surfaces which contain an algebraic system of ∞^1 cubics which do not constitute a pencil, and discusses some fundamental properties of these surfaces.

4. The differential equation of the axis curves of a given conjugate system on a surface, and the conditions for their indeterminateness, were given by Professor Wilczynski in a paper

which appeared in the *Transactions*, volume 16. Making use of these conditions, Mr. Lane considers the problem of finding those surfaces on which axis curves are indeterminate. He reduces the problem to the integration of two ordinary linear homogeneous differential equations of the third order, and obtains a parametric representation for the coordinates of an arbitrary point on the most general surface of this kind. When two cones with a common vertex are given, a surface of the kind under consideration is determined, except for a three-parameter group of projective transformations, which leave invariant the common vertex of the two cones and every straight line through this point. The fundamental conjugate system on the surface then consists of the two one-parameter families of plane curves cut out on the surface by the tangent planes of the two cones. A number of geometric theorems concerning certain closely allied curves and surfaces are obtained.

5. This paper by Mr. Borden deals with the elementary theory of the mixed equation

$$(1)\quad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0$$

along lines initiated by Poisson (*Journal de l'Ecole Polytechnique*, tome 6 (1806), pages 127–141), and in intimate relation to the analogous theory of the Laplace equation

$$s + ap + bq + cz = 0$$

as developed for instance in Forsyth's "Theory of Differential Equations," volume 6, pages 44–96.

The mixed equation has two fundamental invariants under the group of transformations

$$f(x) = v(x)g(x).$$

If these invariants are properly chosen, the equation can be solved in finite form if one of them is zero. The equation is unchanged in form by the Laplace-Poisson transformations

$$(S)\qquad f_{S_1}(x) = f(x+1) + p(x)f(x),$$

$$(T)\qquad f_{T_1}(x) = f'(x) + q(x-1)f(x),$$

so that if the equation resulting from one or more applications

of (*S*) or (*T*) to (1) has a zero invariant it may be solved in finite form. The solution of (1) is then obtained by a recurrence relation. The necessary and sufficient conditions that the above zero invariant exists are obtained.

More general transformations analogous to the one used by Lévy (*Journal de l'Ecole Polytechnique*, tome 38 (1886), page 67) are investigated and are found not to be generally useful in obtaining solutions.

6. The general class of integrals

$$J(x) = \int_0^\infty \varphi(t) \frac{g(x+t)}{g(x)}\, dt,$$

in which

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

includes several instances each of which is central in integral transformations of fundamental importance. Professor Carmichael points out that the fundamental convergence properties of these integrals are of simple character when $\varphi(t)$ and $g(x)$ are subject to certain broad general restrictions. Under these restrictions he develops the basic general elements of this convergence theory.

7. In this note Professor Lefschetz gives an account of a study of abelian manifolds from the point of view of their connectivity and their multiple integrals.

The results arrived at are closely related to some general theorems recently published by him along that line and also to the classical investigations on hyperelliptic surfaces of Picard, Humbert, and the geometers of the Italian school.

8. In a paper presented to the Society at the February meeting Dr. Rider developed a theory of the calculus of variations for n dimensions, using an integral of the form

$$\int_{t_0}^{t_1} f(x_1, \cdots, x_n, \tau_1, \cdots, \tau_{n-1}) \sqrt{x_1'^2 + \cdots + x_n'^2}\, dt.$$

The present paper gives the f_1-function for such an integral. It will be published in *Washington University Studies*.

9. Professor Wilczynski's paper involves a discussion of the various rational osculants determined by a given analytic function, the geometric relations between their zeros and poles, and a number of new notions closely related to intrinsic geometry.

10. In this note Mr. Rouse obtains a number of equivalent sets of conditions for linear dependence relative to linear integral forms of the type

$$\varphi_1(x) - \int_a^b k_1(x, y)\varphi_1(y)dy - \int_a^b k_2(x, y)\varphi_2(y)dy \equiv L(x)$$

which are analogous to the conditions for linear dependence which one finds in connection with the theory of linear algebraic equations. These conditions can easily be extended to the case of a system of m linear integral expressions in n functions.

11. It is known that the determinant D of order n whose n^2 elements are independent variables is invariant under a group G of order $(n!)^2$, and hence this polynomial assumes $n^2!/(n!)^2$ values when it is transformed by the substitutions of the symmetric group of degree n^2. These values may be arranged in pairs such that each pair is composed of polynomials which differ only with respect to sign and hence the square of D is transformed into itself by a substitution group K of order $2(n!)^2$. The main object of Professor Miller's paper is to study the groups G and K.

The groups G and K are simply isomorphic with imprimitive groups of degree $2n$ corresponding to the permutation of the rows and columns of D. When $n > 3$, G is a simply transitive primitive group and the subgroup composed of all its substitutions which omit a given element has two transitive constituents. The group $\underline{G}$ corresponding to the permutations of the positive terms of D under G is also a simply transitive primitive group but its subgroup composed of all the substitutions which omit a given element has a number of transitive constituents which increases indefinitely with n. The factors of composition of G are 2, 2, $n!/2$, $n!/2$, and hence the factors of composition of K are 2, 2, 2, $n!/2$, $n!/2$ and both G and K are unsolvable whenever $n > 3$.

12. The object of Professor Miller's second paper is to furnish a group theory proof for the following two well known theorems: Each of the sums obtained by adding $p-1$, $p-2$, $\cdots$, 2 figurate numbers of orders 2, 3, $\cdots$, $p-1$ respectively is divisible by p whenever p is any odd prime number. Each of the coefficients except the first and the last in the expansion of $(a+b)^p$ is divisible by p. It is not claimed that the group theory proofs are easier than those usually given but they are based upon essentially different concepts and exhibit another element of contact between group theory and elementary number theory. The new proofs are based upon properties of the Sylow groups of order p^{p+1} contained in the symmetric group of degree p^2.

13. In this paper Professor Brand solves the systems composed of the differential equation of the elastic curve of a beam,

$$EIy'' = -M(x),$$

and the homogeneous boundary conditions $y(0)=0, y'(0)=0$ and $y(0)=0, y(l)=0$ by the use of Green's functions; he obtains two modified forms of the Fraenkel formula for flexural deflections which are particularly adapted to the study of statically indeterminate beams. After forming the solutions of the above equation with the non-homogeneous conditions $y(0)=\alpha, y'(0)=\alpha'$ and $y(0)=\alpha, y(l)=\beta$, the results are applied to determine the moments and reactions at the supports, and the deflections, of restrained and continuous beams under any system of loads. In particular, a very general form of Clapeyrou's theorem of three moments is obtained. The usual form of the Fraenkel formula, which has hitherto been deduced from the principles of mechanics, is shown to be a consequence of the above differential equation and its boundary conditions.

14. In this paper Professor Emch proves the theorem that every affinity in space, a collineation leaving the plane at infinity invariant, and defined in cartesian coordinates by

$$x' = a_0 + a_1x + a_2y + a_3z,$$
$$y' = b_0 + b_1x + b_2y + b_3z,$$
$$z' = c_0 + c_1x + c_2y + c_3z,$$

may be represented as the product of a rotation, a similitude, and a perspective affinity, or a homology with the center of homology at infinity. This is still true when the affinity is singular, so that all points (x, y, z) are transformed into points (x', y', z') of a certain plane. In this case the perspective affinity becomes a parallel projection. Based upon this theorem, it is not difficult to prove Pohlke's theorem and a number of related propositions. The demonstration of the theorem stated above, also for the case of a singular affinity, is necessary for a complete proof of Pohlke's theorem by this method.

The paper will appear in the *American Journal of Mathematics*.

15. Dr. Schweitzer constructs 10 classes of systems of postulates for a group, finite or infinite. The classes have the notations $\Sigma[\phi(x, y)], \Sigma[f(x, y)], \Sigma[\psi(x, y)], \Sigma[\theta(x, y)], \Sigma[\phi(x, y), f(x, y)]$, etc., where concretely $\phi(x, y) = xy$, $f(x, y) = xy^{-1}$, $\psi(x, y) = x^{-1}y$, $\theta(x, y) = x^{-1}y^{-1}$. To each of these undefined relations corresponds a fundamental iterative relation which may or may not be "self-conjugate"; in particular the relation for $\theta(x, y)$ is $\theta\{x, \theta(y, x)\} = \theta\{\theta(x, y), x\} = y$, which is self-conjugate. A commutator and transform satisfy certain quasi-transitive iterative functional equations. The preceding iterative relations are generalized. Iterative relations when the group is Abelian are also given. When the group is finite, among other sets of postulates two systems $\Sigma(xy^{-1})$, $\Sigma(x^{-1}y)$ are constructed analogous to Weber's system for a finite group and correspondingly two semi-groups "conjugate" to one another are found. A system $\Sigma[\theta(x, y)]$ contains the restriction that there exists uniquely an element x_0 such that $\theta(x_0, x_0) = x_0$. Under the class $\Sigma[\theta(x, y)]$ an analogue of the semi-group based on Weber's system is the "pseudo-group" with the properties (aside from unique closure) $\theta[\theta(x, y), x] = \theta[x, \theta(y, x)] = y$; $\theta(x', y) = \theta(x, y)$ implies $x' = x$; $\theta(x, y') = \theta(x, y)$ implies $y = y'$. Corresponding to the preceding systems of postulates for group, semi-group, pseudo-group, the author constructs postulational theories of fields. Illustrations from the theory of complex numbers are given.

16. The aim of Dr. Schweitzer's second paper is to describe in general logical terms the standard concepts of mathematical

independence. The author discriminates between an "invariant" and a "variable" type of independence of propositional functions $P_1, P_2, \cdots, P_n$ $(n = 2, 3, \cdots)$ with variable components $R_1, R_2, \cdots, R_i$ $(i = 1, 2, \cdots)$ of respective ranges $\rho_1, \rho_2, \cdots, \rho_i$. A second discrimination is that between "formal" and "material" independence. Subordinate to these discriminations is the characterization of the grade of independence by an index (n, k) $n \geqq k \geqq 0$ where k is the number of false propositions which arise when specific selections are suitably made from the variables $R_1, R_2, \cdots, R_i$. In this classification Professor E. H. Moore's "complete independence" appears as a variable independence of index (n, n) whereas the independence of Peano appears as an invariant independence of index $(n, 1)$.

In the second part of the paper the author calls attention to the use of "supernumerary" (non-independent) indefinables in primitive propositions as an instrument in the solution of problems of complete independence, i. e., if it is required to construct a completely independent set of postulates for a given logical domain the author would propose as a method for attaining this end a set of postulates involving supernumerary indefinables. Application to the author's postulates for an abstract group is made.

17. The paper of Professor Hildebrandt considers a number of generalizations of the theorem of Toeplitz which gives a necessary and sufficient condition under which if $\text{limit}_n\, x_n = x$ then

$$\text{limit}_n\, y_n = \text{limit}_n\, \Sigma_1^n a_{nm}\, x_m$$

is also x. These generalizations are a result of replacing the finite sum by an infinite sum, and considering under what conditions the sequence $\{y_n\}$ shall approach a limit, when the members of the sequence $\{x_n\}$ constitute a function belonging to various classes in the space of a denumerably infinite number of dimensions. The case in which x is a function of a continuous variable is also considered, the transformations from x to y being then on the basis of integration.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

A THEOREM ON THE VARIATION OF A FUNCTION.

BY DR. PAUL R. RIDER.

THE following is a well known theorem of differential geometry:

The differential quotient $d\phi/ds$ (ds is the element of arc) of a function $\phi(u, v)$ at a point on a surface varies in value with the direction from the point. It equals zero in the direction tangent to the curve $\phi = c$, and attains its greatest absolute value in the direction normal to this curve.*

This theorem admits of a generalization if we use a more comprehensive definition of length, a definition sometimes employed in the calculus of variations. Let then

$$S = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

be the generalized length of arc along a curve

$$(C) \qquad x = x(t), \quad y = y(t).$$

By reason of homogeneity conditions†

$$S = \int_{t_0}^{t_1} F(x, y, \cos\theta, \sin\theta)\sqrt{x'^2 + y'^2}dt$$

$$= \int_{s_0}^{s_1} F(x, y, \cos\theta, \sin\theta)ds,$$

in which

$$\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

Then

$$A = \left|\frac{d\phi}{dS}\right| = e\frac{\phi_x dx + \phi_y dy}{F(x, y, \cos\theta, \sin\theta)ds}$$

$$= e\frac{\phi_x \cos\theta + \phi_y \sin\theta}{F(x, y, \cos\theta, \sin\theta)},$$

* See Eisenhart, Differential Geometry, pp. 82–83.

† See Bolza, Vorlesungen über Variationsrechnung, p. 194.

where subscripts indicate partial differentiation, and where e is chosen equal to ± 1 so as to make A positive. Differentiating A with respect to θ, and setting the result equal to zero, we get

$$(1)\quad F(-\phi_x \sin\theta + \phi_y \cos\theta) - (\phi_x \cos\theta + \phi_y \sin\theta)(-F_{x'} \sin\theta + F_{y'} \cos\theta) = 0,$$

$F_{x'}$, $F_{y'}$ denoting partial derivatives of F with respect to its third and fourth arguments respectively. Since

$$F = F_{x'} \cos\theta + F_{y'} \sin\theta,^*$$

equation (1) reduces to

$$(2)\quad \phi_y(x, y)F_{x'}(x, y, \cos\theta, \sin\theta) - \phi_x(x, y)F_{y'}(x, y, \cos\theta, \sin\theta) = 0,$$

and if we define direction on the curve $\phi = c$ by means of the angle $\bar{\theta} = \arctan(-\phi_x/\phi_y)$, (2) becomes

$$F_{x'}(x, y, \cos\theta, \sin\theta)\cos\bar{\theta} - F_{y'}(x, y, \cos\theta, \sin\theta)\sin\bar{\theta} = 0.$$

But this equation determines the value of θ to which the curve $\phi = c$ is transversal.†

Therefore the differential quotient $d\phi/dS$ is equal to zero in the direction tangent to the curve $\phi = c$ and has its maximum absolute value in the direction to which the curve $\phi = c$ is transversal.

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TANGENTIAL INTERPOLATION OF ORDINATES AMONG AREAS.

BY DR. C. H. FORSYTH.

(Read before the American Mathematical Society December 27, 1917.)

IF we wish to interpolate several values in each interval between the successive ordinates $u_0, u_1, u_2, \cdots, u_n$ by finite differences, only a low order of differences can with propriety be used, since high orders based on ordinary statistical data

* See Bolza, loc. cit., p. 196.
† See Bolza, loc. cit., p. 303.

lead to errors. But low orders of differences mean that few of the ordinates can be used at a time, and hence the curve passing through the values intermediate to u_1 and u_2 is not in general identical with the curve passing through the values intermediate to u_2 and u_3, and similarly for the rest of the intervals. It follows that in such successive interpolations the final and complete series of interpolated values have discontinuities at their points of intersection u_1, u_2, etc., which are sometimes quite serious.

Osculatory interpolation represents an attempt to eliminate these discontinuities and succeeds for all practical purposes. It requires that whatever interpolation formula may be used as a basis, two intersecting interpolation curves passing through interpolated values in two adjacent intervals shall have the same slope and curvature at their point of intersection. Thus, the two curves are said to have a common osculating circle at this point and the modification of the basic interpolation formula is called the corresponding osculatory interpolation formula.

In recent years the most prominent statisticians have come to agree that differences higher than third or fourth are inappropriate for interpolating among ordinary statistical data and, as osculatory interpolation requires the use of fifth differences, formulas have been derived ignoring the idea of curvature and reducing the order required to third differences. However, in order to give full credit to the original plan even these third difference formulas are often referred to as "osculatory"; it has seemed more appropriate to the writer to refer to such formulas as "tangential" and the term will be used with that meaning in what follows.

In the December, 1916, issue of the *Quarterly Publications of the American Statistical Association* the writer gave the derivation of an interpolation formula which is to be used—explained geometrically—for interpolating ordinates among areas. For example, it may be used to estimate the population for individual ages from populations given in age groups. The formula to second differences is as follows:

$$(1)\quad u_{x/t} = \frac{w_0}{t} + (2x - t + 1)\frac{\Delta w_0}{t^{\cdot}2!} + \{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\}\frac{\Delta^2 w_0}{t^{\cdot}3!}.$$

Thus, the individual value $u_{x/t}$ may be found from groups of t individual values—w_0, w_1, w_2, etc.—and their differences, where

$$w_0 = u_{0/t} + u_{1/t} + \cdots + u_{(t-1)/t}$$

and in general

$$w_x = u_{x+0/t} + u_{x+1/t} + \cdots + u_{x+(t-1)/t}.$$

The purpose of this paper is to modify formula (1) to give a "tangential" interpolation formula.

We assume the desired equation to be of the form

$$u_{(t+x)/t} = u_1 + ax + bx^2 + cx^3 + dx^4, \tag{2}$$

which evidently passes through the value u_1 for $x = 0$.

For $x = t$

$$u_2 = u_1 + at + bt^2 + ct^3 + dt^4.$$

Expressing u_1 and u_2 by formula (1), we have

$$u_2 - u_1 = at + bt^2 + ct^3 + dt^4 = \frac{\Delta w_0}{t} + (1 + t)\frac{\Delta^2 w_0}{2t^2}. \tag{A}$$

The derivative of (2) with respect to x becomes

$$D_x u_{(t+x)/t} = a + 2bx + 3cx^2 + 4dx^3, \tag{3}$$

and for $x = 0$

$$D_x u_1 = a. \tag{4}$$

But from formula (1)

$$D_x u_{x/t} = \frac{\Delta w_0}{t^2} + (2x - 2t + 1)\frac{\Delta^2 w_0}{2t^3}, \tag{5}$$

and for $x = t$

$$D_x u_1 = \frac{\Delta w_0}{t^2} + \frac{\Delta^2 w_0}{2t^3}. \tag{6}$$

Hence, by (4) and (6)

$$a = \frac{\Delta w_0}{t^2} + \frac{\Delta^2 w_0}{2t^3}. \tag{B}$$

Formula (5) may be expressed also in terms of the differences of u_1; hence, as in (6)

$$(7) \qquad D_x u_2 = \frac{\Delta w_1}{t^2} + \frac{\Delta^2 w_1}{2t^3}$$

$$= \frac{\Delta w_0}{t^2} + (2t+1)\frac{\Delta^2 w_0}{2t^3} + \frac{\Delta^3 w_0}{2t^3},$$

since

$$\Delta w_1 = \Delta w_0 + \Delta^2 w_0$$

and

$$\Delta^2 w_1 = \Delta^2 w_0 + \Delta^3 w_0.$$

Formula (3) for $x = t$ becomes

$$(8) \qquad D_x u_2 = a + 2bt + 3ct^2 + 4dt^3.$$

Hence, from (7) and (8)

$$(C) \quad a + 2bt + 3ct^2 + 4dt^3 = \frac{\Delta w_0}{t^2} + (2t+1)\frac{\Delta^2 w_0}{2t^3} + \frac{\Delta^3 w_0}{2t^3}.$$

We now have three equations (A), (B), and (C) out of four necessary equations to determine the values of the four coefficients a, b, c, and d in equation (2). Ordinarily, an equation of the third degree would be sufficient, in which case only three coefficients would be needed, but formula (1) is peculiar in the respect that when a group, say w_1, is broken up into the individual values u_1, $u_{1+1/t}$, $\cdots$ $u_{1+(t-1)/t}$ their sum must finally equal w_1 and in establishing formula (2) this requirement must be maintained. This requirement furnishes the desired fourth equation. If we give x successively the values $0, 1, 2, \cdots, t-1$ in formula (2) and sum the results we obtain w_1 or $w_0 + \Delta w_0$. This yields

$$w_0 + \Delta w_0 = tu_1 + \frac{t(t-1)}{2}a + \frac{t(t-1)(2t-1)}{6}b$$

$$+ \frac{t^2(t-1)^2}{4}c + \frac{t(t-1)(2t-1)(3t^2-3t-1)}{30}d.$$

Transposing the term tu_1 to the left side and noting that by formula (1)

$$u_1 = \frac{w_0}{t} + (t+1)\frac{\Delta w_0}{2t^2} + (1-t^2)\frac{\Delta^2 w_0}{6t^3}$$

we have, on simplifying

$$(D)\quad a+\frac{(2t-1)}{3}b+\frac{t(t-1)}{2}c+\frac{(2t-1)(3t^2-3t-1)}{15}d = \frac{\Delta w_0}{t^2}+(t+1)\frac{\Delta^2 w_0}{3t^3}.$$

Solving the four equations (A), (B), (C), and (D) simultaneously, we obtain

$$a=\frac{\Delta w_0}{t^2}+\frac{\Delta^2 w_0}{2t^3},$$

$$b=\frac{\Delta^2 w_0}{2t^3}+\frac{3t^2-2}{4t^4(t^2+1)}\Delta^3 w_0,$$

$$c=\frac{1-4t^2}{2t^5(t^2+1)}\Delta^3 w_0,$$

$$d=\frac{5\Delta^3 w_0}{4t^4(t^2+1)}.$$

Finally, substituting these values in equation (2) and collecting terms we obtain

$$(9)\quad u_{(t+x)/t}=w+(2x+t+1)\frac{\Delta w}{2}+(3x^2+3x+1-t^2)\frac{\Delta^2 w}{6} + x^2(x-t)(5tx+2-3t^2)\frac{\Delta^3 w}{4t(t^2+1)},$$

where, for brevity, we let $\Delta^n w=\Delta^n w_0/t^{n+1}$ (and $w=w_0/t$).

In applying (9) to, say the group w_1, it is to be noticed that the slope at u_1 $(=u_{t/t})$—by (6)—is determined by w_0, w_1, and w_2 (to be referred to for the present as "the first set") and at u_2 $(=u_{2t/t})$—by (8)—by w_1, w_2, and w_3 (to be referred to as "the second set"). Then, in applying the formula to the next group w_2, its first set (used to determine the slope at u_2) is identical with the second set of the preceding group and the interpolation curves of the two groups have the same slope at their point of intersection u_2; and similarly for u_3, u_4, etc. The idea of curvature could be added if desired in exactly the same way if higher differences were proper.

When, as often happens, $t = 5$, (9) becomes

$$(10)\quad u_{(5+x)/5} = w + (x+3)\Delta w + (x^2 + x - 8)\frac{\Delta^2 w}{2} + x^2(x-5)(25x-73)\frac{\Delta^3 w}{(104)5^2}.$$

It is seldom desirable to use either (1) or (9) to determine a single individual value—they are generally used to break up a group of values completely. In the latter case, those familiar with the use of finite differences know that much labor can be saved by using the corresponding formulas for the leading term and differences of the individual values. The application of such formulas is much shorter on the average than the repeated application of either (1) or (9) and the majority of the computation consists in mere addition.

If we let x equal successively $0, 1, \cdots, t-1$ in formula (9) and difference the results four times we obtain the following leading term and differences

$$u_1 = w + \frac{t+1}{2}\Delta w + \frac{1-t^2}{6}\Delta^2 w,$$

$$\delta u_1 = \Delta w + \Delta^2 w + (1-t)(2+5t-3t^2)\frac{\Delta^3 w}{4t(t^2+1)},$$

$$\delta^2 u_1 = \Delta^2 w + \{(2-3t^2)(6-2t) + 5t(14-6t)\}\frac{\Delta^3 w}{4t(t^2+1)},$$

$$\delta^3 u_1 = \{6(2-3t^2) + 30t(6-t)\}\frac{\Delta^3 w}{4t(t^2+1)},$$

$$\delta^4 u_1 = t\ 5!\Delta^3 w,$$

where, it is to be remembered, $\Delta^n w = \Delta^n w_0/t^{n+1}$ (and $w = w_0/t$).

When t has the value 5 the third differences prove relatively difficult to compute owing to the presence of a "13" in the denominator. If in the difference δu_1, $2.7\Delta^3 w/13$ is replaced by $2.6\Delta^3 w/13$ and in all the rest $\Delta^3 w/13$ is replaced by $\Delta^3 w/12$ the leading term and differences (for $t = 5$) become

$$\begin{aligned} u_1 &= w + 3\Delta w - 4\Delta^2 w, \\ \delta u_1 &= \Delta w + \Delta^2 w + .2\Delta^3 w, \\ \delta^2 u_1 &= \Delta^2 w - .2\Delta^3 w, \\ \delta^3 u_1 &= -.6\Delta^3 w, \\ \delta^4 u_1 &= +.002\Delta^3 w_0, \end{aligned}$$

where, it is to be remembered, $\Delta^n w = \Delta^n w_0/5^{n+1}$ (and $w = w_0/5$). It should be noticed, however, that $\delta^4 u_1$ is expressed more simply by $.002\Delta^3 w_0$.

The arbitrary changes just suggested give an error of only about one half of one per cent of $\Delta^3 w_0$ in w_1 (the sum of the individual values interpolated). The error in each individual value would then be much less and if only two more decimal places are used than are to be finally retained—which are all that are ordinarily necessary as found by experience—the errors would not appear at all in the results. As a check upon the work the sum of the individual values interpolated should be w_1 as given originally.

It should be pointed out that the formulas derived above can not be used for "end" values; that is, if the groups of values were $w_0, w_1, w_2, \cdots, w_n$ the formulas could not be used for interpolating or breaking up w_0 or w_n, for the derivation of the formulas is based upon the use of four values (w_0, w_1, w_2, and w_3) to break up w_1; that is, there must always be at least one group preceding the group to be broken up. To break up "end" values formula (1) could be used or the formulas for the leading term and differences to be found in the article cited.

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NON-EUCLIDEAN GEOMETRY.

Geometrical Researches on the Theory of Parallels. By N. LOBACHEVSKI. Translated from the original by G. B. HALSTED. New edition. Chicago and London, Open Court, 1914. 8vo. 50 pages. Cloth, price $1.25.

NON-EUCLIDEAN geometry had two independent discoverers: Johann Bolyai (1802–1860), a Hungarian officer in the Austrian army; and Nicolaus Lobachevsky (1793–1856), son of a Russian peasant, and graduate, professor, and rector, of the University of Kasan.*

As early as 1823 the former had grasped the real nature of his problem, and in 1829 he sent a completed manuscript on the subject to his father, Wolfgang Bolyai, who was a

* The best history of non-euclidean geometry is by Bonola. It was translated into English and edited by H. S. Carslaw (Chicago, 1912).

professor of mathematics, physics and chemistry. In 1832 the father published a Latin translation of the manuscript as an appendix* to the first volume of one of his scientific works.

It seems to have been after 1823 that the development of non-euclidean geometry occurred to Lobachevsky. This may be inferred from a passage in his book on Elementary Geometry:† "Up to the present no one has been able to find a rigorous proof of the fifth postulate of Euclid; what has been given we may only call explanations; these do not deserve to be characterized as mathematical proofs in the true sense." The first fruit of new study in a geometry independent of Euclid's hypothesis was a paper read before the physico-mathematical section of the University of Kasan in 1826. This paper does not seem to have been preserved, but the essential parts of it were contained in a memoir published in 1829–30. Lobachevsky indicated further applications of the new theory in three more Russian memoirs of 1835, 1835–38, 1836 ("Géométrie Imaginaire" which appeared in *Crelle's Journal* for 1837 is really a free translation of the memoir of 1835). He endeavored to call the attention of mathematicians to his discoveries by publishing a summary of his results in a little book which appeared at Berlin in 1840 under the title: Geometrische Untersuchungen zur Theorie der Parallellinien.‡ It is this little work which Dr. Halsted has here given us in English form.

It is probable that Lobachevsky never even heard of Bolyai, who, in 1848, made himself thoroughly acquainted with the Geometrische Untersuchungen.

* A facsimile edition was published at Maros-Vásárhelyini in 1907. 4 + 28 pp.

† The manuscript of this work was sent to St. Petersburg in 1823 but failed of publication at that time on account of the unfavorable report of the academician N. Fuss. In 1898 the manuscript was discovered in the archives of the university of Kasan and it was published in 1909 by the Physico-Mathematical Society of Kasan (8vo. 3 + 67 pp.). Titles of sections indicate that Méray's ideas of fusion were anticipated, many years before, by Lobachevsky: Measuring lines; Angles; Perpendiculars; Measuring solid angles; Regular polygons and polyhedrons; Equivalent triangles; Measuring right angles, triangles and other figures; Parallelograms; Measuring of pyramids and polyhedrons; Measuring the circumference and the area of a circle, the volume of the cylinder and cone; Measuring the area of the right cylinder and right cone; On the magnitude of the volume and surface of the sphere.

‡ Facsimile reprint, Berlin, 1887; [Geometrical works of Lobachevsky] (in Russian), Kasan, vol. II, 1886, pp. 553–578. French translation by J. Hoüel in *Mém. de Bordeaux*, tome 4 (1866), also separately (2d ed. 1895), and in Recherches géométriques sur la théorie des parallèles, Paris, 1900.

There is little doubt that Carl Friedrich Gauss (1777–1855) was the first to have a clear view of a geometry independent of the fifth postulate; but this fact "was only revealed after the works of Lobachevsky (1829–30) and J. Bolyai (1832) appeared." To these geniuses alone, by reason of priority of publication, must belong all honor due to those confident of their results and unafraid of the world's criticism.

In February, 1832, Wolfgang Bolyai sent a copy of his son's work to Gauss. In acknowledging its receipt Gauss wrote, on March 6,* somewhat as follows:

". . . Now something concerning the work by your son. If I commence with this, '*that I can not praise it*,' you may well be taken aback for a moment, but I can not do otherwise; to praise it would be to praise myself: for the whole content of the work, the path which your son has followed, and the results to which he is led, agree almost throughout with my own meditations which have occupied my mind for 30–35 years. So I am surprised to the utmost. My intention with respect to my own work, concerning which little has been put on paper up to the present, was to allow nothing to be known of it during my life time. Most people do not have any appreciation of what is involved in the subject, and I have found only few men who took up with any particular interest what I communicated to them concerning it. To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what is lacking, and upon this matter almost all men are far from clear. On the other hand, it was my intention some time to put the whole matter on paper, so that it should not one day perish with me.

"Greatly have I been surprised that I am spared this trouble and much do I rejoice that it is just the son of my old friend who has got the start of me in such a remarkable manner."

And then follow a discussion of details and a presentation of his own development of some of the parts.

In a letter to Gerling about three weeks earlier Gauss characterized the younger Bolyai as a "genius of the first order." He wrote as follows:

". . . Noch bemerke ich, dass ich dieser Tage eine kleine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle *meine eigenen Ideen und RESULTATE* wiederfinde, mit grosser Eleganz entwickelt,

* Gauss, Werke, Bd. 8, Göttingen, 1900, pp. 220–221.

obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein *sehr* junger österreichischer Officier, Sohn eines Jugendfreundes von mir. . . . Ich halte diesen jungen Geometer v. BOLYAI für ein Genie erster Grösse. . . ."

Gauss was not only familiar with all the writings of Lobachevsky referred to above but he possessed all of them by 1844, and indeed, probably a year or so earlier, when he was responsible for having Lobachevsky elected as correspondent of the Royal Society of Sciences at Göttingen. Contrasting the Geometrische Untersuchungen with the memoirs, he wrote to Gerling on February 8, 1844:* "Namentlich hat sie viel mehr Concinnität und Präcision, als die grösseren Aufsätze des Lobatschewsky, die mehr einem verworrenen Walde gleichen, durch den es, ohne alle Bäume erst einzeln kennen gelernt zu haben, schwer ist, einen Durchgang und Uebersicht zu finden."

In a letter to his friend Schumacher in 1846 Gauss wrote further† concerning the Untersuchungen: ". . . materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte Art in ächt geometrischem Geiste. Ich glaube Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird. . . ."

The ideas inaugurated by Lobachevsky and Bolyai did not obtain wide recognition for many years, indeed for more than a decade after Gauss's death. Shortly after Hoüel had published in 1866 the French translation of the Untersuchungen and "Appendix" together with the commendatory passages from Gauss's correspondence, the works of the almost unknown Slav and Hungarian mathematicians came into their own, and non-euclidean geometry was taken up as a serious study.

The work before us for review is described as a "new edition." The translation seems to have been originally published,‡ with "Translator's preface," at Austin, Texas, in *Scientiæ Baccalaureus* for February, 1891.§ A new edition

* Gauss, Werke, Bd. 8, p. 237.

† Gauss, Werke, Bd. 8, p. 238.

‡ D. M. Y. Sommerville (Bibliography of Non-Euclidean Geometry, 1911) lists an edition published at Rolla, Mo., in 1891.

§ Vol. 1, no. 3, pp. 123–164.

with type reset, with the same preface now dated May 1, 1891, with translator's introduction and with translator's appendix was published as a Bulletin of the University of Texas.* This same edition with date 1891 and with a "Note to the fourth edition," was apparently published in 1892. But "The Nemonic Series, Vol. IV" is also called the fourth edition though listed by Sommerville as the third. Halsted's translation was also published in the *Tōkyō Sūgaku Butsurigaku Kwai Kiji* (Proceedings of the Physico-mathematical society of Tokyo).†

Except as to binding, the edition before us is practically identical with that issued by the University of Texas. Indeed the same plates seem to have been used except for pages 5, 9 and 10. Small changes have been made on pages 7, 19, 47, 48. Pages 48 and 49 of the translator's preface have been replaced by paragraphs headed "Bibliography" and the portrait frontispiece of Lobachevsky, which has been added, seems to have been reproduced from Engel's volume.‡

It seems a pity that a little more care had not been exercised in this final revision. On page 6 we still find Henricus Billingsly instead of Henry Billingsley. While the following sentence may have been true in 1891, the date of the preface, it does not describe the conditions in 1914, the date of the volume. "Even to-day in the vast system of examination set by Cambridge, Oxford and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms." Figures 13, 15, 24, 28, 31, 32, 33, still differ in some respects from Lobachevsky's originals. For example these had: lines EG and EH in Fig. 13; no arc $C'B'$ in Fig. 15; LK as bisector of HC in Fig. 24 and so on.

The translation is faithfully and skilfully done. While the paragraphing is quite different from the original the change has possibly contributed to clearer presentation. The long and unnecessary spacings after sections 16 and 24 do not occur in the original.

The Open Court publishing company has rendered a service

* Austin, 1891, 50 pp.

† Vol. 5 (1892–94), pp. 6–50.

‡ Nikolaj Iwanowitsch Lobatschefskij. Zwei geometrische Abhandlungen aus dem Russischen uebersetzt, mit Anmerkungen und mit einer Biographie des Verfassers. Von F. Engel. Leipzig, 1898. Reviewed by F. S. Woods in this BULLETIN, vol. 6, 1900, pp. 339–344; 443–452.

to mathematicians in placing at their disposal English editions of Lobachevsky's and Bolyai's masterpieces.

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FRENCH SCIENCE.

Science and Learning in France with a Survey of Opportunities for American Students in French Universities. An Appreciation by American Scholars. The Society for American Fellowships in French Universities, 1917. 8vo. 40 + 454 pages. (For sale by A. C. McClurg and Co., Chicago: cloth, $1.50; stiff paper, $1.00.)

Exposition Universelle et Internationale de San Francisco. La Science Française. 2 tomes. Paris, Ministère de l'Instruction Publique et des Beaux Arts, 1915. 397 + 405 pages.

L'Histoire des Sciences et les Prétentions de la Science Allemande. By E. PICARD. Paris, Perrin et Cie., 1916. 12mo. 49 pages. Boards. Price 60 centimes.

THE Society for American Fellowships in French Universities has announced that it is prepared to award ten or more fellowships, each of one thousand dollars annually, to American graduate students who undertake to study in French Universities. The fellowships will be given for studies in any of the following fields of knowledge: anthropology, archæology, astronomy, botany (and agriculture), chemistry, criminology, education, engineering, geography, geology, history, law, mathematics, medicine, classical philology, oriental philology, romance philology, semitic philology, English philology, philosophy, physics, political science (including economics and international law), psychology, religion, sociology, and zoology (including biology).

The awards will be based on nation-wide competition. It is planned to make the first annual award in the spring of 1918, so that the Fellow may depart for France in July, 1918, if the war has ended by that time. It has been announced that a circular of information, giving details as to the persons eligible, the form of application, and the mode of making the award, will be ready for distribution to all American institu-

tions by October, 1917. Each fellowship is to be held for two years, and new fellowships will be awarded each year for ten years at least.

Preparatory to putting this scheme into effect the Society has published a sumptuous handbook for American graduate students in French universities. This is the work under review. It was edited by Professor John H. Wigmore, of Northwestern University.

In the Introduction Charles W. Eliot writes on "The Mind of France," and George E. Hale, a correspondent of the Institute of France, tells of "The Intellectual Inspiration of Paris." Then follow 33 chapters, each prepared by a drafting committee of specialists and each illustrated by a portrait of at least one Frenchman distinguished in the subject treated. "Each chapter," to quote from the preface, "sets forth briefly

"1. The progress of French science and learning in the particular field during the last half century or so; the notable achievements; the eminent leaders; the special lines of development; in general, the share in the world's accomplishments;

"2. The courses of instruction given, now or recently, at the universities of France, particularly at the University of Paris; the names of the most important scholars, with a mention of their principal works and of the special fields of research over which they preside;

"3. The facilities available for study and research, including libraries, laboratories, archives and museums, the auxiliary institutes, special schools, and learned societies and committees.

"There is also an Appendix describing the methods of instruction and study, the standards of preparation expected of the student, the system of degrees, the customs as to residence, attendance, etc.; the regulations as to fees, etc.; and other facts useful to the visiting student." The appendix is illustrated by pictures of buildings in the University of Paris, etc.

The drafting committee for the chapter on Astronomy consisted of: P. Fox, G. E. Hale, F. R. Moulton, W. D. MacMillan, and H. N. Russell; for the one on Engineering: I. N. Hollis, H. M. Howe, A. C. Humphreys and A. Sauveur; for the one on Mathematics: D. R. Curtiss, T. F. Holgate, E. H. Moore, and E. B. Wilson; and for the chapter on Physics: H. Crew, A. A. Michelson, and W. C. Sabine.

The committee on mathematics finds that this science "has always made a special appeal to French genius distinguished by its fondness for logic and its striving for perfection in form," and it pays tribute to the genius of Henri Poincaré, of whom "at his death in 1912 it was the universal verdict that he must be considered the greatest mathematician of his age." The portrait of Poincaré reproduced in the chapter was taken about thirty years ago. It is the same one that was reproduced in the *American Journal of Mathematics* for 1890.

A sentence on page 357 seems to indicate that the writer does not know that the Sorbonne is only that part of the University of Paris in which instruction is given in the faculties of sciences and letters.

The volume is of extraordinary interest.

On account of the war, France's exhibit at the San Francisco exposition of her scientific achievements was of an unusual nature. It consisted of a library in which were collected periodicals and books, ancient and modern, illustrating her scientific progress and activity. "Pour chaque science," President Poincaré writes in the introduction, "Pour chaque science, on a essayé de remonter au moment où, en France, un ordre d'études, importantes par le profit intellectuel ou moral qu'elles procurèrent aux hommes, fut abordé pour la première fois et devint l'objet de recherches systématiquement conduites. On a voulu marquer l'origine, le point d'où sont partis tant de hardis explorateurs pour l'éternel voyage à la recherche de la vérité; on a indiqué, sur les chemins tracés par leurs glorieux efforts, les sommets d'où ont été aperçus de nouveaux horizons; on a signalé enfin, avec quelque insistance, l'étape actuellement atteinte qui sera dépassée par le travail de demain poursuivi dans des direction que l'on a cherché à préciser.

"On verra donc dans la bibliothèque des livres vénérables et illustres par où une grande idée fut semée dans le monde; puis les publications principales, grâce auxquelles les rameaux puissants d'une doctrine se développèrent; enfin, pour l'heure présente, un choix assez large d'ouvrages individuels ou de recueils collectifs où l'on trouvera la preuve tangible de l'activité scientifique de la France, et où l'on pourra apprécier la luxuriante floraison produite par une habile culture."

In order that the visitor might better appreciate the choice of works, two volumes of admirable articles were written by

distinguished specialists to summarize France's contributions to each science. These articles were each concluded by a bibliography, which is quite extensive in some cases.* Each is illustrated by an excellent portrait of some deceased master in the field considered.

The 14 articles in the first volume are on: philosophy (by Bergson, 23 pages), sociology, science of education, mathematics (by Appell, 15 pages), astronomy (by Baillaud, 37 pages), physics (by Bouty, 21 pages), chemistry, mineralogy, geology, paleobotany, zoological paleontology, biology, the medical sciences, the science of geography. The 19 articles of the second volume deal particularly with the sciences of archaeology, history, art, language, and philology.

In the article on mathematics there are six main headings: arithmetic, algebra, mathematical analysis, geometry, kinematics and mechanics. Under astronomy there are three parts, one on the seventeenth century, a second on the eighteenth and the first part of the nineteenth century, the third on contemporary astronomy.

The volumes should be in every college library.

The motto for Picard's little book is by Etienne Lamy, secrétaire perpétuel de l'Académie française. It is as follows:

"Les Allemands ont pris pour devise: L'Allemagne au-dessus de tout.

"Nous ne répondons pas: La France au-dessus de tout.

"Une seule devise est digne de la France: Au-dessus de tout, la vérité."

Picard confines his attention to the mathematical, physical and natural sciences. He shows that few indeed are the contributions of Germans toward their development, in comparison with those by scholars of other countries. While an attempt is made to explain the psychology of the German's "au-dessus de tout" illusion, the great discoveries of such men as Gauss are not slighted by any means. As far as it goes, the survey is very satisfying.

But the intellectual triumphs of France along these lines are much more fully set forth in the volumes whose plans have been described above. One of them, happily, it has been America's privilege to prepare. R. C. ARCHIBALD.

BROWN UNIVERSITY, *July*, 1917.

* In the article on mineralogy, for example, about fifteen pages. Books whose titles are marked with an asterisk in these bibliographies were to be seen in the library exhibit.

FUNCTIONS OF TWO COMPLEX VARIABLES.

Lectures Introductory to the Theory of Functions of Two Complex Variables. By A. R. FORSYTH. Cambridge University Press, 1914. xvi + 281 pp.

THE present volume consists substantially of a course of lectures delivered in the University of Calcutta in January and February, 1913, in response to a special invitation of the authorities accompanied by a stipulation that the lectures should be published. What was desired was an exposition of some subject that might suggest openings to those who had the will and the skill to pursue research. In accordance with this wish the author selected the theory of functions of two complex variables, a subject still in a preliminary stage of its development and one into the exposition of which he could incorporate a considerable body of results of his own.

No attempt is made to give a systematic discussion of the whole subject nor is attention concentrated upon one particular issue. Several distinct lines of investigation are dealt with, even though this required that their treatment should be relatively brief. The essential purpose throughout was to deal with a selection of principles and of generalities belonging to the initial stages of the theory of functions of two complex variables; and this was accompanied by the desire to establish some new results and to suggest some new problems of investigation.

The substantial results of the theory of functions of a single complex variable are assumed to be so familiar to the reader that only brief and indirect reference to them will usually suffice. Almost everywhere in the exposition the number of independent variables is restricted to two. Many of the propositions may readily be modified so as to apply to the case of n variables; but this is not always true. Consequently we have on the one hand a range of results which belong essentially to functions of more than one variable and on the other hand another range of results belonging essentially to functions of just two variables. Typical of the latter is the theory of quadruply periodic functions of two variables.

At several places the author has departed from the usual custom of dealing with only a single function of two complex

variables and has considered simultaneously two such functions. Certain characteristic properties are in this way brought into fuller discussion and interesting results emerge which otherwise would not be obtained, particularly in the theory of quadruply periodic functions of two variables. He is led to the simultaneous investigation of two functions by the following considerations (pages 2–4): In the theory of functions w of a single variable z we have a relation of the form $f(w, z) = 0$ so that we may consider w as a function of z or z as a function of w, thus having a complete dual notion of inversion. But if w is a function of two complex variables z and z' we do not have such inversion in the complete sense. Consequently a second function w' is introduced. Then we have relations

$$F(w, w', z, z') = 0, \qquad G(w, w', z, z') = 0,$$

so that we may look upon w and w' as functions of z and z' or conversely upon z and z' as functions of w and w'.

This way of introducing the duality certainly leaves something still to be desired. There is nothing in the forms of the foregoing equations to suggest the pairing of w and w' on the one hand and of z and z' on the other. Moreover, if we are interested primarily in a single function of two variables we certainly have an added difficulty when we introduce another essentially unrelated function of the same two variables. It can hardly be expected that two such functions will throw essential light each upon the properties of the other. Only certain more general characteristics of a function can be expected to emerge in this way. This undoubtedly is valuable, but it also leaves something further to be desired in the way of information about the individual function primarily in consideration.

A modification of the method employed by the author seems to obviate both of these difficulties. Instead of considering only two essentially unrelated functions w and w' of the variables z and z', let us also treat in further detail the following special case, namely:

$$w = f(z, z'), \quad w' = f(z', z).$$

Under easily derived appropriate conditions which we shall not take space to state, it is clear that z and z' are inversely

functions of w and w' of the form

$$z = f_1(w, w'), \quad z' = f_1(w', w).$$

We thus retain the complete duality of inversion which is so important in the case of functions of a single variable; and this is accomplished without the introduction of an essentially new function. Moreover, the pairs w, w' and z, z' play a rôle different from that of other pairs of pairs of the four quantities w, w', z, z'. All the results obtained by considering the two functions w and w' state properties belonging essentially to w alone, since w and w' are so simply related.

The first chapter deals principally with three methods of geometrical representation of two complex variables, namely: representation by points in four-dimensional space, representation by lines in ordinary three-dimensional space, representation by points in two planes. The conclusion is reached (page 14) that only two of these are even fairly useful, namely, the first and the last. To supply fully our needs in respect of geometric representation we must have some uniquely effective new idea. While awaiting its appearance we must be content with such imperfect representations as the above-named methods afford.

In the first chapter one finds also a definition of functions of two complex variables similar to the Riemann definition for the case of one variable.

In the second chapter we have a treatment of the linear transformation of complex variables:

$$w = \frac{az + bz' + c}{a''z + b''z' + c''}, \quad w' = \frac{a'z + b'z' + c'}{a''z + b''z' + c''}.$$

Such a transformation is reducible to one or another of three canonical forms according as the characteristic equation in θ, namely,

$$\begin{vmatrix} a - \theta & b & c \\ a' & b' - \theta & c' \\ a'' & b'' & c'' - \theta \end{vmatrix} = 0,$$

has three simple roots, one double root and one simple root, or one triple root. Invariant centers of the transformation are discussed. A special investigation is also given of the sort of

frontiers in four-dimensional space which are left invariant as to quality by these transformations, the result corresponding to invariancy of circles under linear transformations in a single variable. If one puts $z = x + iy$, $z' = x' + iy'$, x, y, x', y' being real variables, then the invariant frontiers in question are defined by equations of the form

$$A(x^2 + y^2) + B(x'^2 + y'^2) + C(xx' + yy') + D(xy' - yx') + Ex + Fy + Gx' + Hy' = K,$$

where the symbols $A, \cdots, K$ denote constants. There is also a treatment of the invariants and covariants of quadratic frontiers, and of periodic transformations.

It seems to the reviewer that a considerable number of readers will be repelled from the book by its first two chapters unless they observe the relation, or rather for the most part the lack of relation, of these two chapters to the remainder of the work. They are in large measure a reproduction of two papers by the author. Their general interest is certainly less than much of that which follows, both of the author's researches and of the matter taken from the literature. Fortunately but little of these chapters is used in building the succeeding portions of the book. Consequently it would seem well to have a list of portions suited to a first reading, similar to such a list in Forsyth's Theory of Functions of a single variable. We venture to suggest such a selection, our primary aim being to indicate to the reader how he may readily get into the meat of the subject.

In the first chapter one may take sections 1–6, 14–16. The second chapter may be omitted entirely on a first reading, or one may take sections 21–27 so as to have in mind the first properties of linear transformations. Thus after reading eleven pages from the first chapter (and possibly eleven pages from the second chapter), one is ready to enter upon the main features of the subject at the beginning of the third chapter on page 57. From this point forward one may read Chapters III, IV, V, Chapter VI up to page 169, Chapter VIII (with reference to one or two parts of Chapter VII). The remainder of the book may then be taken in any order desired. It appears to the reviewer that the reader is likely to find the book more interesting if he follows a plan of selection similar to that just indicated.

In the third chapter begins the essential development of the theory of single-valued analytic functions of two complex variables. The problem here treated is that of their expressibility in power series. Certain of the fundamental theorems for functions of a single variable are readily carried over to the case of functions of two variables; as, for instance, those connected with the Taylor and the Laurent series expansions, dominant functions, approach to every value, and certain matters concerning analytic continuation. But in the classification of singularities an essential difference emerges. There are three types of singularities; to these are given the names *pole, unessential singularity, essential singularity.* Let k, k' be a singular place for the single-valued function $f(z, z')$. If no power series $P_0(z-k, z'-k')$ in non-negative powers of $z-k, z'-k'$ exists such that the product

$$P_0(z-k, z'-k')\cdot f(z, \quad z')$$

is expressible in a series of non-negative powers of $z-k$, $z'-k'$, then k, k' is called an essential singularity of $f(z, z')$. If the foregoing product can be formed so as to afford a function $P_1(z-k, z'-k')$ expansible in non-negative powers of $z-k, z'-k'$, then $f(z, z')$ is said to have a pole or an unessential singularity at k, k' according as it is or is not possible to choose $P_0(z-k, z'-k')$ so that $P_1(0, 0)$ shall be different from zero.

The fourth chapter is devoted principally to a consideration of the classic theorems of Weierstrass concerning the representation of a single-valued function in the vicinity of any of its various places, whether ordinary or singular. But little is said of the case of an essential singularity. For all other places the matter rests on the representation of a function in the vicinity of a zero of the function, as one sees readily from the definitions stated in the preceding paragraph. The concept of the divisibility of one function by another and the consequent theory of reducibility are also treated in this chapter. On pages 83 and 84 the author implicitly makes essential use of certain results in the theory of divisibility, whereas the theory is itself first developed on pages 112 ff.

In Chapter V one meets the following (and related) theorems analogous to corresponding ones for functions of a single variable: A function without essential singularities (either in

the finite region or at infinity) is a rational function; a function which has no essential singularity in the finite region is expressible as a quotient of two functions each of which is without any singularity in the finite region.

The sixth chapter is devoted to integrals. In the first part is given a treatment of double integrals of uniform functions of two variables. Here one finds Poincaré's extension of Cauchy's main integral theorem, followed by several simple examples of a subject which awaits further development. The latter part of the chapter is concerned with integrals, both single and double, of algebraic functions, a theory to which Picard has made fundamental contributions. The author here takes the line of introducing two algebraic functions of two variables, thus following up (for the first time except for his treatment of linear transformations) his notion of the simultaneous consideration of two functions. But little more is done than to give an introduction to this problem, particularly in its preliminary formal aspects. Reference is made to Picard and Simart's Théorie des Fonctions algébriques de deux Variables indépendantes, where the case of a single function of two variables is treated in illuminating detail. The opinion is expressed that the further simultaneous consideration of two functions will lead to interesting extensions of the theory.

The seventh chapter deals with the so-called "level places" of two simultaneous single-valued functions of two variables, each function being without an essential singularity in the finite region. It is shown that two such functions simultaneously approach zero for a suitable approach of z and z' separately to suitably chosen points in their respective planes of variation, these planes being understood to contain the point infinity. Such a point is called a common zero of the two functions. If the two functions are further restricted to be independent and without common factors, then their common zeros are isolated. These results are readily applied to a consideration of the "level places" of two functions $f(z, z')$ and $g(z, z')$, namely, the common zeros of $f(z, z') - \alpha$ and $g(z, z') - \beta$, where α and β are constants.

The eighth and last chapter is devoted to the theory of single-valued periodic functions of essentially two variables. Such a function cannot have more than four linearly independent period-pairs. The cases of one period-pair and of

two period-pairs are degenerate. Type forms of period-pairs for the non-degenerate cases of three and of four period-pairs are worked out in detail. The method is somewhat tedious and probably could not be extended to the case of functions of n variables; but it brings into relief the character of the various cases which initially come into consideration. The theta-functions of two variables are employed in the development of the theory. Theorems concerning algebraic relations among homo-periodic functions are derived. Simple examples of hyperelliptic functions are introduced. But the author has not attempted to expound the details of the theory of periodic functions of two variables; he has left that to the specific treatises on this subject.

A few minor misprints may be mentioned: "real" for "pure imaginary" in line 15 of page 32; "not universally" for "indeed never" in line 16 up on page 83; obvious misprints in footnote on page 75, in line 16 up on page 85, in line 15 on page 126, in line 5 up on page 236.

On the whole the book is well written and the exposition is usually clear. Some (but not many) infelicities of expression occur of which the following are examples: The author speaks several times (as on page 50) of "invariance under a single transformation only"; he employs repeatedly the redundant description "uniform, continuous and analytic" (see an instance on page 91 and the definition of "analytic" on page 59); he speaks sometimes (as on page 60) of a fixed domain as if it might be infinitesimal in extent. On page 241 and elsewhere in the following discussion it is desirable to specify *uniform* convergence as well as *absolute* convergence.

It is desirable to have an explicit definition of the region at infinity; this is not supplied by the author. It is assumed without remark (page 58 and elsewhere) that the infinite region in the z-plane and also that in the z'-plane is the infinite region of the usual complex plane; and then the infinite region for z and z' simultaneously is that in which either z or z' is infinite.

Probably minor defects of this sort (of which we do not attempt a complete list) will interfere with nothing but the comfort of the reader likely to be interested in this subject and will not seriously impair for him the value of the book.

We add a remark concerning a choice deliberately made by the author, a choice particularly unfortunate in the opinion of the reviewer.

Consider the function $e^{1/x}$. For $x = 1$ this function has the value e. When x approaches the value 1 in any way the function approaches the value e. We may say that the function possesses or acquires the value e at the point $x = 1$. Now consider what happens to the function as x approaches zero. It is easy to show that the function approaches e if x approaches zero *in a proper manner*. It is also equally easy to show that the function approaches any preassigned value if x approaches zero *in a suitable manner*. We may then say that $e^{1/x}$ *approaches* e (or any other assigned value) in case x approaches zero in a way appropriately dependent on this assigned value. Forsyth would still speak of the value e as a value which the function *acquires* or *possesses* at the point $x = 0$, using precisely the same terms as for the case of x approaching 1 and deliberately rejecting the distinction which clearly exists. See especially pages 77 and 78.

To the reviewer it seems definitely desirable to make the distinction and to mark it by a suitable choice of terminology, using *acquires* or *possesses* in the one case and *approaches* in the other. Some such distinction is generally made. There is a definite loss and no gain in refusing to make it, as most readers will probably be led to suspect by an examination of Forsyth's remarks (pages 77–79) about Picard's theorem to the effect that a non-constant entire function $f(z)$ acquires every value with at most a single exception.

Many of the results stated in the book need to be interpreted in the light of the fact that the author refuses to make the distinction in question. We illustrate by a single example. On page 198 we have the theorem: "Two independent functions, regular throughout the finite part of the field of variation, vanish simultaneously at some place or places within the whole field." Here "vanish" means to "acquire" the value zero in the sense of Forsyth's use of the term. What is actually proved is that the two functions simultaneously approach zero for an appropriate approach of z, z' to some appropriately chosen place k, k', finite or at infinity. The theorem does not look so attractive when clothed in this dress; but its true character is more readily recognized.

Similar remarks may be made about not a few other theorems. It is unfortunate for the reader to have to look through the proof of a theorem to see which sense of "acquire" gives to the theorem the best or largest meaning which the proof allows.

Finally, let us say that there can be no doubt that Forsyth has rendered his colleagues a distinct service in adding this book to his already long list of useful publications. It will be of definite value to a large number of persons interested in the theory of functions of complex variables.

R. D. CARMICHAEL.

SHORTER NOTICES.

The Geometrical Lectures of Isaac Barrow. Translated, with notes and proofs, by J. M. CHILD, B.A. (Cantab.), B.Sc. (Lond.). The Open Court Publishing Company, Chicago, 1916. xiv+218 pp.

"ISAAC BARROW was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673."

"The above is the ultimate conclusion that I have arrived at, as the result of six months' close study of a single book, my first essay in historical research. By the 'Infinitesimal Calculus,' I intend 'a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations.'"

These are the opening paragraphs of the preface to this edition of Barrow's Lectures. While the rest of the book does perhaps not justify the claims of the preface, it furnishes a very welcome addition to the generally available information concerning Barrow. It presents, in abridged form, a translation by Mr. Child of the "Lectiones Geometricæ" of 1670, of which a first English translation was published by Edmond Stone in 1735. Numerous notes, bearing upon Mr. Child's thesis, are scattered throughout the text, proofs have been added in a number of places, and there is an introduction of

32 pages. There we find a brief account of the life and work of Barrow, of the work on infinitesimal calculus, done before his time and of the mutual influence of Barrow and Newton; a description of the original text, an argument to show that Barrow obtained his results analytically, although he published them in geometrical form, and finally a list of the analytical equivalents of his chief theorems.

Mr. Child appears to base his conclusions on the following points: (1) Barrow gave in his "Lectiones Geometricæ" a complete treatment of the elements of both the differential and integral calculus, including the fundamental theorem; (2) he obtained his results by means of some analytical device, essentially the same as that used by Newton; (3) the essential parts of the "Lectiones Geometricæ" were completed before 1664, the year in which "Barrow first came into close personal contact with Newton"; (4) Leibniz bought a copy of Barrow's work in 1673. The present reviewer believes that Mr. Child's work definitely establishes the first of these points and furnishes strong arguments for the second and third, without, however, settling them absolutely. Additional arguments for the conclusion concerning Leibniz are reserved for the future.

It seems not at all unlikely therefore that we shall have to place Barrow at least on a par with, if not above, Newton and Leibniz among the inventors of the calculus. Just where the Japanese mathematician Seki, whose "circle principle" antedates the work of Barrow, should be placed is a matter to be decided by historians who are concerned with questions of priority and with the influence of one mind upon another. To them I must also leave it to judge in how far Mr. Child's conclusions are justified by the evidence he has so far presented.

But entirely apart from their bearing upon historical questions, Barrow's lectures are of interest to the modern reader. In the first five lectures are developed various properties of curves, which are thought of kinematically, viz., as generated by a point descending with increasing velocity along a line which moves uniformly and parallel to itself. A tangent to a curve is spoken of as a line which has one point in common with the curve and which lies entirely on one side of it. In his fourth lecture, Barrow then shows that the slope of the tangent is equal to the slope of the curve at the point of tangency. He is thereby led to conceive of the tangent as

"a prolongation of an element of the curve," which is "to be considered as composed of an infinite succession of infinitesimal straight lines."

The remaining seven lectures are devoted to a purely geometrical treatment of problems of tangents and areas, not, as was the case in the work of most of Barrow's predecessors, for special curves, but for general classes of curves. One of the characteristics of Barrow's method, and a consequence of its geometrical character, is the constant use of what we now know in our textbooks as the "subtangent" and "polar subtangent," and it is somewhat of a surprise to find these good friends outside a textbook. So we have in Lecture VIII the theorem, that if $\varphi(x) = a/f(x)$, then the subtangents for points with the same abscissa on the curves $y = f(x)$ and $y = \varphi(x)$ are equal but opposite in sign. In Lecture IX it is shown that if $[\varphi(x)]^m = a^{m-n}[f(x)]^n$, then the subtangents for corresponding points on the curves $y = \varphi(x)$ and $y = f(x)$ are in the ratio $m : n$; i. e., Barrow derives a formula for the differentiation of a fractional power. Each theorem on curves in cartesian coordinates is followed by a corresponding one dealing with curves given in polar coordinates. There are given formulas for the differentiation of the sum, the product and the quotient of two functions, for the differentiation and integration of $\tan\theta$, of $\sqrt{x^2 + a^2}$, etc. Let us close by quoting two theorems, appearing in Lectures X and XI respectively and constituting together the essence of the fundamental theorem of the calculus: "Let ZGE be any curve of which the axis is AD; and let ordinates applied to this axis, AZ, PG, DE, continually increase from the initial ordinate AZ; also let AIF be a line such that, if any straight line EDF is drawn perpendicular to AD, cutting the curves in the points E, F, and AD in D, the rectangle contained by DF and a given length R is equal to the intercepted space $ADEZ$; also let $DE : DF = R : DT$, and join DT. Then TF will touch the curve AIF"; and "Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, $TF : FM = R : FZ$. Then the space $ADLK$ is equal to the rectangle contained by R and DB."

ARNOLD DRESDEN.

Die mathematischen Grundlagen der Variations- und Vererbungslehre. Von P. RIEBESELL. Leipzig, B. G. Teubner, 1916. 45 pp.

THIS little book is volume 34 of the Mathematische Bibliothek, published under the editorship of Lietzmann and Witting. It is doubtless intended for the biological, rather than for the mathematical public, although its title might lead one to expect a somewhat more fundamental analysis of the larger mathematical questions which arise in modern biological research. The first 32 pages are devoted to an exposition of the elements of the theory of probabilities and of the problems connected with the determination of a function that will represent a given distribution of statistical data. To make the task of reading the book as easy as possible for the non-mathematical reader, the theory is followed step by step by numerical examples. In the last 9 pages there is found a discussion of some of the biological questions in which the mathematical problems play a rôle. Probably the greatest use that one may expect from a book like this would come if it should stimulate some mathematician to acquaint himself further with the mathematical problems that arise in biological theory or if it should lead a biologist to more extended mathematical studies. If two sciences are to cooperate effectively, there must be trained "agents de liaison," who will make possible a partial overlapping of the spheres of interest.

ARNOLD DRESDEN.

Methoden zur Lösung geometrischer Aufgaben. Von B. KERST. (Mathematische Bibliothek herausgegeben von W. Lietzmann und A. Witting, Band 26.) Leipzig, Teubner, 1916. 12mo. 2 + 47 pages. Paper, price M.0.80.

AMONG the scores of books, pamphlets and articles which have been written concerning methods of solving problems of elementary synthetic geometry, the work of the late Julius Petersen, the Dane, is easily the best by reason of its elegant exposition, comprehensiveness, and suggestiveness. The first edition was published in 1866;* but the enlarged second edition, of which English and German translations were published in the same year, 1879, is the one whose contents are most familiar.†

* Methoder og Theorier til Løsning af geometriske Konstruktionsopgaver, anvendte paa c. 300 Opgaver. Kjøbenhavn, 1866. 4 + 85 pp. + 1 pl.

† There have been at least 18 editions of this remarkable work: 6 in

In his "Vorwort" Kerst states: "Von der benutzten Literatur ist zu erwähnen: Tropfke, Geschichte der Elementar-Mathematik, 1903; Petersen, Methoden und Theorien zur Auflösung geometrischer Konstruktionsaufgaben, Kopenhagen, 1879 (im Buchhandel nicht mehr zu haben); R. Von Fischer-Benzon, Programm Kiel Gymnasium, 1884. Einige Aufgaben entstammen dem Aufgaben-Repertorium aus Hoffmanns Zeitschrift für math. u. naturw. Unterricht."

When it is recalled that Tropfke's work is wholly historical and that the part of Fischer-Benzon's work dealing with "method" is acknowledged to be largely due to Petersen, we would expect to find here much which has been made familiar by our old friend. In this expectation we are not disappointed. All of the methods and most of the problems are in Petersen's work.

It is, then, hardly necessary to indicate at length the nature of the contents of Kerst's little book except to say that it contains 17 paragraphs each developing a certain method, for the most part by the solution of problems. In all, there are seventy of the problems given with solutions, and about as many more to solve.

In presenting a solution of the problem of Apollonius by means of Petersen's parallel translation of circles, Kerst makes the unwarranted statement (page 47): "Das ist wahrscheinlich auch der Weg, auf dem Apollonius selbst zur Lösung gelangte."

There is probably no English reader in this country who would ever turn to this work in preference to that by Petersen. But of course the latter costs six times as much. It is doubtless significant of the dearth, or cost, of German supplies about two years ago that this number of the Mathematische Bibliothek appeared in paper covers instead of with the cloth back and board covers which had become familiar to us in connection with the first twenty volumes of the series.

R. C. ARCHIBALD.

Danish, 5 in French, 2 in Russian and one each in Dutch, English, German, Hungarian, and Italian. The English edition is readily obtainable from Copenhagen, and the French edition from Paris.

NOTES.

AT the meeting of the London mathematical society held March 14, 1918, the following papers were read: By G. H. HARDY, "The representation of a number as the sum of any number of squares"; by G. N. WATSON, "A problem in the theory of numbers."

THE following university courses in mathematics are announced:

CORNELL UNIVERSITY (academic year 1918–1919).—By Professor JAMES MCMAHON: Theory of probabilities, three hours; Introduction to actuarial science, three hours.—By Professor J. H. TANNER: Introduction to the mathematics of finance, two hours.—By Professor VIRGIL SNYDER: Descriptive geometry, three hours (first term); Analytic geometry of space, three hours (second term).—By Professor F. R. SHARPE: Hydrodynamics, three hours (first term); Elasticity, three hours (second term).—By Professor W. B. CARVER: Projective geometry, three hours.—By Professor ARTHUR RANUM: Line geometry, three hours (second term).—By Professor D. C. GILLESPIE: Differential equations, three hours.—By Professor W. A. HURWITZ: Differential equations of mathematical physics, three hours.—By Professor C. F. CRAIG: Functions of a complex variable, three hours.—By Professor F. W. OWENS: Advanced calculus, three hours.—By Dr. M. G. GABA: Problems in mathematics, three hours.

JOHNS HOPKINS UNIVERSITY (academic year 1918–1919).—By Professor FRANK MORLEY: Higher geometry, three hours (first term); Theory of functions, three hours (second term); Dynamics and hydrodynamics, two hours (second term).—By Professor A. B. COBLE: Theory of correspondences, two hours.—By Professor ABRAHAM COHEN: Elementary theory of functions, two hours; Applied mathematics, two hours (second term).

UNIVERSITY OF CALIFORNIA (summer session, June 24–August 3).—In addition to the courses offered at Los Angeles (see BULLETIN, this volume, page 363), the following advanced

courses are to be given at Berkeley: By Professor D. N. LEHMER: Theory of numbers, five hours.—By Professor P. J. DANIELL (of the Rice Institute): Differential equations, five hours; Theory of integral equations, five hours.—By Dr. B. A. BERNSTEIN: Functions of a complex variable, five hours.

UNIVERSITY OF CHICAGO. Autumn quarter:—By Professor E. H. MOORE: Matrices in general analysis, three hours.—By Professor L. E. DICKSON: Theory of numbers, three hours.—By Professor E. J. WILCZYNSKI: Projective differential geometry, three hours.—By Professor H. E. SLAUGHT: Differential equations, three hours.—By Professor A. C. LUNN: Heat and molecular physics, three hours; Electron theory, three hours.—Winter quarter:—By Professor E. H. MOORE: Functions of infinitely many variables, three hours.—By Professor L. E. DICKSON: Algebraic numbers, three hours.—By Professor G. A. BLISS: Definite integrals, three hours; Differential equations, three hours.—By Professor E. J. WILCZYNSKI: Projective differential geometry, II, three hours.—By Professor A. C. LUNN: Thermodynamics, three hours; Theory of sound, three hours.—Spring quarter:—By Professor E. H. MOORE: Functions of infinitely many variables, II, three hours; Limits and series, three hours.—By Professor L. E. DICKSON: Linear algebra, three hours; Solid analytics, three hours.—By Professor G. A. BLISS: Functions of lines, three hours; Partial differential equations, three hours; Functions of a complex variable, three hours.—By Professor A. C. LUNN: Geometric optics, three hours.

UNIVERSITY OF ILLINOIS (academic year 1918–1919).—By Professor E. J. TOWNSEND: Functions of a complex variable, three hours; Differential equations and advanced calculus, three hours.—By Professor G. A. MILLER: Continuous groups, three hours (second term); Theory of equations, three hours (first term).—By Professor H. L. RIETZ: Actuarial theory, three hours.—By Professor J. STEBBINS: Least squares, two hours (first term).—By Professor J. B. SHAW: Fundamental functions, three hours (first term); Functional transformations, three hours (second term).—By Professor C. H. SISAM: Invariants and higher plane curves, three hours; Solid analytic geometry, three hours (second term).—By Professor R. D. CARMICHAEL: Elliptic functions, three hours.

—By Professor ARNOLD EMCH: Projective geometry, three hours.—By Professor A. R. CRATHORNE: Calculus of variations, three hours.—By Dr. E. B. LYTLE: History of mathematics, two hours (second term).—By Dr. G. A. WAHLIN: Theory of numbers, three hours.

PROFESSOR GABRIEL KOENIGS has been elected member of the Paris academy of sciences.

PROFESSOR G. CASTELNUOVO, of the University of Rome, has been elected national member of the Reale Accademia dei Lincei; Professors E. BOREL and E. GOURSAT, of the University of Paris, Professor J. HADAMARD, of the Collège de France, and Professor H. LAMB, of the University of Manchester, have been elected foreign associates.

THE Turin academy has elected the following corresponding members: Professors L. BERZOLARI, of the University of Pavia; R. MARCOLONGO, of the University of Naples; L. PINCHERLE, of the University of Bologna; G. RICCI and F. SEVERI, of the University of Padua; G. ALBENGA, G. COLOMETTI and G. A. MAGGI, of the University of Pisa; V. REINA, of the University of Rome.

DR. E. G. TOGLIATTI has been appointed docent in projective and descriptive geometry at the University of Turin.

DR. A. VERGERIO has been appointed docent in analysis at the University of Bologna.

DR. E. G. BILL, assistant professor of mathematics at Dartmouth College, has been granted leave of absence to serve on the staff of the Military Service Council at Ottawa, Canada.

AT the University of California, Professor FLORIAN CAJORI, of Colorado College, has been appointed professor of the history of mathematics. Associate professor D. N. LEHMER has been promoted to a full professorship of mathematics. Dr. B. A. BERNSTEIN, Dr. THOMAS BUCK, and Dr. FRANK IRWIN have been promoted to assistant professorships. Professor G. C. EDWARDS has been made professor emeritus, after forty-five years of service.

ASSISTANT professor ANNA J. PELL, of Mount Holyoke College, has been appointed associate professor of mathematics at Bryn Mawr College. Dr. OLIVE C. HAZLETT, of Bryn Mawr, has been appointed assistant professor of mathematics at Mount Holyoke.

ASSISTANT professor H. M. SHOWMAN, of the Colorado School of Mines, has been promoted to a professorship of mechanics and engineering.

MR. H. L. OLSON has been appointed instructor in mathematics at Heidelberg University, Tiffin, Ohio.

DR. J. V. DEPORTE, of the New York State College at Albany, has been promoted to an assistant professorship of mathematics.

PROFESSOR E. E. LEVI, of the University of Genoa, was killed in battle October 28, 1917, at the age of 34 years. The Italian society of sciences (the XL) awarded him its gold medal in 1912.

PROFESSOR A. VITERBI, of the University of Pavia, was killed in battle November 18, 1917, at the age of 44 years.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, list of 78 titles, including 15 in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). Cambridge papers. London, Macmillan, 1918. 8vo. 6+326 pp. 6s.

BERICHTE und Mitteilungen veranlasst durch die Internationale Mathematische Unterrichtskommission. 2te Folge, III: Titel und Inhaltsübersicht zur ersten und zweiten Folge der Berichte und Mitteilungen. Leipzig, Teubner, 1917. 8vo. 16+99 pp. M. 4.00

GAUJA (P.). Les fondations de l'Académie des Sciences (1881–1915), avec une préface par A. Lacroix et E. Picard, secrétaires perpetuels de l'Académie des Sciences. Paris, Gauthier-Villars, 1917. 8vo. 6+658 pp. Fr. 12.00

KOENIGSBERGER (L.). Weierstrass' erste Vorlesung über die Theorie der elliptischen Funktionen. Leipzig, Teubner, 1917. 8vo. 32 pp.

LACROIX (A.). See GAUJA (P.).

PICARD (E.). See GAUJA (P.).

VISCARDINI (M.). Nota su alcune corrispondenze duali di carattere proiettivo, metrico e statico sulla sfera e sul piano. Roma, Stabilimento tipo-litografico del Genio Civile, 1918. 12 pp. 2 tavole.

II. ELEMENTARY MATHEMATICS.

ANDERSON (R. F.). See HALLETT (G. H.).

BASSI (A.). Esercizi e problemi di algebra complementare. 2a edizione riveduta. Volume 1 (per la terza classe degli istituti tecnici). Parte 2. Livorno, Giusti, 1918. 32mo. 8+210 pp. L. 2.10

BOSTON SCHOOL COMMITTEE. Outline of work in mathematics for intermediate classes in elementary grades. (School documents No. 12.) Boston, School Committee, 1916. 8vo. 9 pp.

HALLETT (G. H.) and ANDERSON (R. F.). Elementary algebra. Boston, Silver-Burdett, 1917. 8vo. 6+402 pp.

KEAL (H. M.) and PHELPS (N. S.). Eighth grade mathematics. Chicago, Atkinson, Mentzer and Company, 1917. 8vo. 5+132 pp. $0.65

——. Secondary mathematics. Book 1. Chicago, Atkinson, Mentzer and Company, 1917. 8vo. 224 pp. $0.80

MENDIZABAL TAMBORREL (J. DE). Tratado de goniometria. 2a edicion. Mexico, Departamento de Talleres Graficos de la Secretaria de Fomento, 1917. 8vo. 209 pp.

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College for the years 1908–1917. Edited by R. M. Milne. London, Macmillan, 1918. 8vo. 7s.

MYERS (G. W.). See PALMER (C. I.).

PALMER (C. I.) and TAYLOR (D. P.). Teachers' handbook for Palmer and Taylor's plane geometry; edited by G. W. Myers. (Lake Mathematical Series.) Chicago, Scott, Foresman and Company, 1917. 8vo. 14+232 pp. $2.50

PHELPS (N. S.). See KEAL (H. M.).

TAYLOR (D. P.). See PALMER (C. I.).

III. APPLIED MATHEMATICS.

ANNUAIRE pour l'an 1918 publié par le Bureau des Longitudes. Avec des notices scientifiques par G. Bigourdan, J. Renaud, M. Hamy et E. Picard. Paris, Gauthier-Villars, 1918. 16mo. 10+870 pp. Fr. 2.00

ARMAGAT (E. H.) et DÉCOMBE (L.). La statique des fluides, la liquéfaction des gaz et l'industrie du froid. 1re et 2e partie. Paris et Liège, Béranger, 1917. 6+265 pp. Fr. 18.00

AUGHTIE (H.). Applied mechanics, second year. London, Routledge, 1918. 8vo. 2s. 6d.

BAILEY (W. B.) and CUMMINGS (J.). Statistics. Chicago, McClurg, 1917. 6+153 pp. Cloth. $0.60

BIGOURDAN (G.). See ANNUAIRE.

BOSTON SCHOOL COMMITTEE. Special syllabus, drawing and manual training; grades 1–8, 1917–1918. (School documents Nos. 13–14.) 2 volumes. Boston, School Committee, 1917. 8vo.

COUSIN (A.). Résumé pratique de navigation. Long cours. Plaisance. Cabotage. 2e édition. Paris, Challamel, 1917. 8vo. Cartonné. Fr. 6.50

CUMMINGS (J.). See BAILEY (W. B.).

DECANTE (E.). Tables d'azimut pour tous les points situés entre les cercles polaires et les astres dont la déclination est compris entre 0° et 48°. 3e édition augmentée. Tome 8: 59° à 66°. Paris, Gauthier-Villars, 1917. 8vo. Fr. 6.00

DÉCOMBE (L.). See ARMAGAT (E. H.).

DUNWOODY (H.). Mechanics, sound, light, thermo-mechanics and hydraulics. London, Chapman and Hall, 1918. 8vo. 13s. 6d.

ELDERTON (W. P.). Frequency curves and correlation. Addendum with diagram and errata. London, Layton, 1917. 22 pp. 1s. 6d.

HAMY (M.). See ANNUAIRE.

HARTMAN (W. B.). Machine shop practice. New York, Appleton, 1917. 10+247 pp. Cloth.

HOSMER (G. L.). Navigation. New York, Wiley, 1918. 12 mo. 9+214 pp. Cloth. $1.25

HUGHES (C. H.). Handbook of ship calculations, construction and operation. A book of reference for ship owners, ship officers, ship and engine draughtsmen, marine engineers and others engaged in the building and operating of ships. New York, Appleton, 1917. 24+740 pp. Leather.

KNIBBS (G. H.). The mathematical theory of population applied to the data of Australian population statistics. Appendix A. Volume 1. Melbourne, McCarron, Bird and Company, 1917. 16+466 pp.

MORITZ (F.). Méthode de transformation des diagrammes de Watt (volume et pression) en diagrammes entropiques par le réseau entropique. Paris, Gauthier-Villars, 1917. 16mo. 10 pp. Fr. 1.80

MUIR (W. C. P.). A treatise on navigation and nautical astronomy including the theory of compass deviations prepared for use as a textbook at the U. S. Naval Academy. Fourth edition revised and enlarged. Annapolis, Md., The United States Naval Institute, 1918. 18+784 pp. $4.20

D'OCAGNE (M.). Principes de la nomographie. Application des nomogrammes à l'alignement aux différents cas de résolution des triangles sphériques. Paris, Gauthier-Villars, 1917. 8vo. 42 pp. Fr. 1.50

PERRY (H.). Theories of energy. New York, Putnam, 1918. 8+231 pp. Cloth. $1.75

PICARD (E.). See ANNUAIRE.

RENAUD (J.). See ANNUAIRE.

THOMAS (A. O.). Rural arithmetic. New York, American Book Company, 1918.